Math 451: Introduction to General Topology
Lecture 21

We now discuss non-permanence of properties under products.

Example (product of two normal isn't normal). Recall that the Sorgenfrey plane $S \times S$, where $S := \mathbb{R}$ is the Sorgenfrey line whose top. is gen. by $[a,b)$, is not normal (optional HW), yet the Sorgenfrey line S is normal because it is regular and Lindelöf.

First ctbly (hence metrizability) are not closed under unctbl products, more precisely:

Prop. Let $X := \prod_{i \in I} X_i$ be a product of unctbly many nontrivial T_1 top. spaces X_i. Then X is isn't 1^{st} ctbl; in fact, no point has a ctbl neighbourhood basis.

Proof. Let $x \in X$ and suppose towards a contradiction that x has a ctbl neighb. basis $\{U_n\}_{n \in \mathbb{N}}$. Since for each $n \in \mathbb{N}$, $\exists$ finite-base cylinder C_n s.t. $x \in C_n \subseteq U_n$, $\{C_n\}_{n \in \mathbb{N}}$ is also a neighbourhood basis of x. Each cylinder C_n has a finite base $I_n \subseteq I$. Then $I' := \bigcup_{n \in \mathbb{N}} I_n$ is still ctbl, while I is unctbl, so $\exists\ i_0 \in I \setminus I'$. Let $a \in X_{i_0}$ be distinct from $x(i_0)$. Because X_{i_0} is T_1, $\exists$ open $U_{i_0} \subseteq X_{i_0}$ with $x(i_0) \in U_{i_0}$ but $a \notin U_{i_0}$. Then $[i_0 \mapsto U_0]$ is an open neighbourhood of x, so $\exists\, n$ s.t. $C_n \subseteq [i_0 \mapsto U_0]$. But the point $y \in X$ defined by $y(i) := \begin{cases} x(i) & i \neq i_0 \\ a & i = i_0 \end{cases}$ belongs to C_n since $i_0 \notin I_n$, but is not in $[i_0 \mapsto U_{i_0}]$, contradicting $C_n \subseteq [i_0 \mapsto U_0]$. $\square$

Compactness (= topological finiteness).

Def. A top. space is called **compact** if every open cover admits a finite subcover.

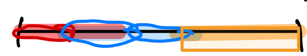
Thus, this a strengthening of Lindelöf property.

Rephrasing compactness via closed sets. A top. space X is compact $\iff$ every collection $\mathcal{C}$ of closed sets with the finite intersection property has a nonempty intersection, i.e.
$$\bigcap \mathcal{C} := \bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$
We say that a collection $\mathcal{C}$ of sets satisfies the finite intersection property (FIP) if every finite $\mathcal{C}_0 \subseteq \mathcal{C}$ has a nonempty intersection, i.e. $\bigcap \mathcal{C}_0 := \bigcap_{C \in \mathcal{C}_0} C \neq \emptyset$.

Proof. $\Rightarrow$. Let's prove the contrapositive: suppose there is a collection $\mathcal{C}$ of closed sets with FIP but $\bigcap \mathcal{C} = \emptyset$. Then let $\mathcal{U} := \{X \setminus C : C \in \mathcal{C}\}$ (called the dual collection). Then $\mathcal{U}$ is an open cover of X because $\emptyset = \bigcap \mathcal{C} \Rightarrow$ (via complements) $X = \bigcup \mathcal{U}$, by deMorgan. Yet $\mathcal{U}$ has no finite subcover: for any finite $\mathcal{U}_0 \subseteq \mathcal{U}$, $\emptyset \neq \bigcap_{U \in \mathcal{U}_0} U^c$ because of FIP, hence (by deMorgan), $X \neq \bigcup_{U \in \mathcal{U}_0} U$.

$\Leftarrow$. If $\mathcal{U}$ is an open cover of X, then the dual of $\mathcal{U}$, namely, $\mathcal{C} := \{X \setminus U : U \in \mathcal{U}\}$ is a collection of closed sets with $\emptyset$ intersection because $X = \bigcup \mathcal{U}$ implies $\emptyset = \bigcap \mathcal{C}$. Thus, $\mathcal{C}$ doesn't have the FIP, i.e. $\exists$ finite $\mathcal{C}_0 \subseteq \mathcal{C}$ with $\bigcap \mathcal{C}_0 = \emptyset$, so (by deMorgan), $\bigcup_{C \in \mathcal{C}_0} (C^c) = X$, so $\mathcal{U}_0 := \{C^c : C \in \mathcal{C}_0\}$ is a finite subcover. $\square$

Def. Let $\mathcal{U}$ be a cover of a set X. Another cover $\mathcal{V}$ of X is called a refinement of $\mathcal{U}$ if every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.



Prop. For every open cover $\mathcal{U}$ of a top. space X, every basis $\mathcal{B}$ of X contains an open cover of X refining $\mathcal{U}$.

Proof. For each $x \in X$ $\exists$ $U \in \mathcal{U}$ covering x, i.e. $x \in U$, so $\exists$ $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$. Then $\mathcal{V} := \{B_x\}_{x \in X}$ is an open cover of X and refines $\mathcal{U}$ by definition. $\square$

Obs. Let X be a top. space. (HW)

(a) X is compact $\iff$ every open cover admits a finite refinement.

(b) X is Lindelöf $\iff$ every open cover admits a ctbl refinement.

Cor. In the def of compactness it suffices to consider open covers consisting of sets in some basis. More precisely, a top. space X is compact $\iff$ for every basis $\mathcal{B}$ of X, every open cover $\mathcal{U} \subseteq \mathcal{B}$ of X has a finite subcover.

Proof. The nontrivial direction is $\Leftarrow$. Let $\mathcal{B}$ be a basis and $\mathcal{U}$ an open cover. Then by above, $\exists$ open cover $\mathcal{V} \subseteq \mathcal{B}$ which refines $\mathcal{U}$. By the hypothesis, $\mathcal{V}$ admits admits a finite subcover $\mathcal{V}_0$, which is hence a finite refinement of $\mathcal{U}$. $\square$

Examples. (a) All finite top. spaces are compact.

(b) In the cofinite top on $\mathbb{N}$, $\mathbb{N}$ is compact. In fact, every subset of $\mathbb{N}$ is compact (in the subspace top). HW

(c) $\mathbb{R}$ is not compact: $\{(n-1, n+1)\}_{n \in \mathbb{Z}}$ is an open cover with no finite subcover. Also, $\{(-n, n)\}_{n \in \mathbb{N}}$ is an open cover with no finite subcover.

The last example generalizes:

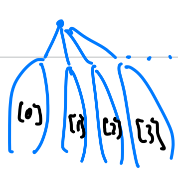
Prop. If a top. space X admits an unbounded compatible metric then X is not compact.

Proof. If d is an unbdd metric on X generating the top., then for any $x_0 \in X$, the collection $\{B_n^d(x_0)\}_{n \in \mathbb{N}}$ is an open cover of X but doesn't have a finite subcover. $\square$

Cor. Compact subsets of metric spaces are bounded.

However boundedness of the metric doesn't guarantee compactness even for complete metric spaces.

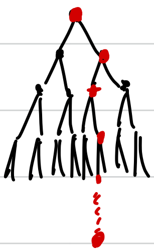
Example. The Baire space $\mathbb{N}^{\mathbb{N}}$ has compatible complete metric bdd by 1 (the usual metric). Yet $\mathbb{N}^{\mathbb{N}}$ is not compact: the collection of cylinders $\{[n]\}_{n \in \mathbb{N}}$ is an open cover with no finite subcover.



The noncompactness occurred because the tree was infinitely branching, in contrast to:

<u>Example.</u> The Cantor space $2^{\mathbb{N}}$ is compact. In fact, $\Sigma^{\mathbb{N}}$ is compact for finite Σ.

Proof. We show for $2^{\mathbb{N}}$ but the proof is the same for any finite Σ.



Let $\mathcal{U}$ be an open cover of $2^{\mathbb{N}}$ and suppose towards a contradiction that it has no finite subcover. Call a vertex $w \in 2^{<\mathbb{N}}$ heavy if $[w]$ still doesn't have a finite subcover of $\mathcal{U}$. By our assumption the empty word (the root) is heavy. Note that if w is heavy then at least one of $w0$ and $w1$ is heavy (since otherwise if $\mathcal{U}_0, \mathcal{U}_1 \subseteq \mathcal{U}$ are finite covers of $[w0]$ and $[w1]$, then $\mathcal{U}_0 \cup \mathcal{U}_1$ is a finite cover of $[w]$). Using this we start from the root and obtain an infinite branch $x \in 2^{\mathbb{N}}$ s.t. $\forall n \in \mathbb{N}$, $x|_n$ is heavy, i.e. $[x|_n]$ doesn't have a finite cover in $\mathcal{U}$. But $\mathcal{U}$ covers x, so $\exists U \in \mathcal{U}$ with $x \in U$, so for a large enough $n \in \mathbb{N}$, $[x|_n] \subseteq U$, but then $\{U\}$ is an open cover of $[x|_n]$ of size 1, a contradiction. □

<u>Remark on the compactness of subspaces.</u> Let X be a top. space and $Y \subseteq X$. An open cover of Y, by definition, is a collection $\mathcal{U}$ of sets $U \subseteq Y$ which are relatively open in Y, i.e. $\exists$ set $\tilde{U} \subseteq X$ open in X s.t. $U = \tilde{U} \cap Y$. For example, $\{[0, \frac{2}{3}), (\frac{1}{2}, 1]\}$ is an open cover of $[0,1]$ in the subspace top. However, we could replace $\mathcal{U}$ with the collection $\tilde{\mathcal{U}} := \{\tilde{U} : U \in \mathcal{U}\}$, so it consists of sets open in X and it still covers Y in the sense that $Y \subseteq \bigcup \tilde{\mathcal{U}} := \bigcup_{\tilde{U} \in \tilde{\mathcal{U}}} \tilde{U}$. Clearly, $\mathcal{U}$ has a finite subcover of $Y \iff \tilde{\mathcal{U}}$ has a finite subcover of Y. Thus:

<u>Obs.</u> For a top. space X, a subspace $Y \subseteq X$ is compact if every cover $\mathcal{V}$ of Y (i.e. $Y \subseteq \bigcup \mathcal{V}$) with sets open in X, there is a finite subcover $\mathcal{V}_0 \subseteq \mathcal{V}$ of Y (i.e. $Y \subseteq \bigcup \mathcal{V}_0$).